Lecture 33

More on moment generating functions

Recall that for a random variable X, the moment generating function of X is defined to be

$$M_X(t) = E[e^{tX}].$$

The mgf has the property that for any $n \geq 1$,

$$M_X^{(n)}(0) = E[X^n], \text{ the } n^{th} \text{ moment.}$$

Ex: Let X be an exponential RV with parameter λ.

Then

$$\begin{aligned} M_X(t) &= E[e^{tX}] \\ &= \int_0^\infty e^{tx} \lambda e^{-\lambda x}\, dx \\ &= \lambda \int_0^\infty e^{-(\lambda - t)x}\, dx \\ &= \frac{-\lambda}{\lambda - t} e^{-(\lambda - t)x} \Big|_0^\infty = \frac{\lambda}{\lambda - t} \end{aligned}$$

$$M_X^{(n)}(t) = \frac{n!\,\lambda}{(\lambda - t)^{n+1}} \Rightarrow E[X^n] = \frac{n!\,\lambda}{\lambda^{n+1}} = \frac{n!}{\lambda^n}$$

Ex: Let $Z = N(0,1)$. Then

$$M_Z(t) = E\left[e^{tZ}\right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x^2 - 2tx)/2}\, dx.$$

complete the square:
$x^2 - 2tx = (x-t)^2 - t^2$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-t)^2/2} e^{t^2/2}\, dx.$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \frac{e^{-(x-t)^2/2}}{\sqrt{2\pi}}\, dx.$$

Now, $f(x) = \dfrac{e^{-(x-t)^2/2}}{\sqrt{2\pi}}$ is the density function of

$N(t,0)$, so $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Hence $M_Z(t) = e^{t^2/2}$.

Now, let $X = N(\mu, \sigma^2)$. Then $X = \mu + \sigma Z$.

Hence

$$M_X(t) = E\left[e^{tX}\right] = E\left[e^{t(\mu + \sigma Z)}\right]$$

$$= E\left[e^{t\mu} e^{t\sigma Z}\right]$$

$$= e^{t\mu} E\left[{}^{t\sigma Z}\right]$$

$$= e^{t\mu} e^{(t\sigma)^2/2}$$

$$M_X(t) = e^{\sigma^2 t^2/2 + \mu t}.$$

Prop: Let X, Y be indep. Then

$$M_{X+Y}(t) = M_X(t) M_Y(t).$$

Pf:
$$M_{X+Y}(t) = E[e^{t(X+Y)}]$$
$$= E[e^{tX} e^{tY}]$$
$$= E[e^{tX}] E[e^{tY}]$$
$$= M_X(t) M_Y(t).$$

Rmk: If $M_X(t)$ exists and is finite in some region around $t=0$, then $M_X(t)$ uniquely determines the distribution of X:

Theorem (uniqueness) Suppose there is $\delta > 0$ s.t.

$$M_X(t) = M_Y(t) < \infty \quad \forall t \in (-\delta, \delta),$$

then $F_X(t) = F_Y(t) \quad \forall t \in \mathbb{R}.$

The proof is beyond the scope of this course, but we encourage you to read it on your own (see: inversion formula for the Fourier transform).

Joint moment generating functions

For any n-random variables $X_1, \ldots, X_n$, we define the joint moment generating function as

$$M(t_1, \ldots, t_n) = E\left[e^{t_1X_1 + \cdots + t_nX_n}\right] \quad (t_1, \ldots, t_n) \in \mathbb{R}^n.$$

- Note that we can recover the moment generating functions of each of the X_i (and hence the distributions of the X_i's) as

$$M_{X_i}(t) = E\left[e^{tX_i}\right] = M(0, 0, \ldots, 0, \underset{i^{\text{th}}\text{ spot}}{t}, 0, 0, \ldots, 0)$$

- As in the single variable case, we can completely recover the joint distribution of $X_1, X_2, \ldots, X_n$ from the joint mgf, though the proof is too advanced for this course.

From this:

<u>Fact:</u> $X_1, \ldots, X_n$ are independent iff

$$M(t_1, \ldots, t_n) = M_{X_1}(t_1) \cdots M_{X_n}(t_n).$$

<u>Some inequalities</u>

<u>Prop (Markov's Inequality)</u> Suppose X is a non-negative RV. Then for any $a > 0$,

$$P(X \geq a) \leq \frac{E[X]}{a}.$$

<u>Pf:</u> Let $a > 0$. Set $I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{o/w.} \end{cases}$

Then I is a RV, and

$$I \leq \frac{X}{a}$$ (since if $X \geq a$, then $\frac{X}{a} \geq 1$. and $I = 1$.

So

$$E[I] \leq \frac{E[X]}{a}.$$

This gives the result, since $E[I] = P(X \geq a)$. $\square$.

Using this, we get:

<u>Prop</u> (Chernoff's Bound): Let X be a RV. Then

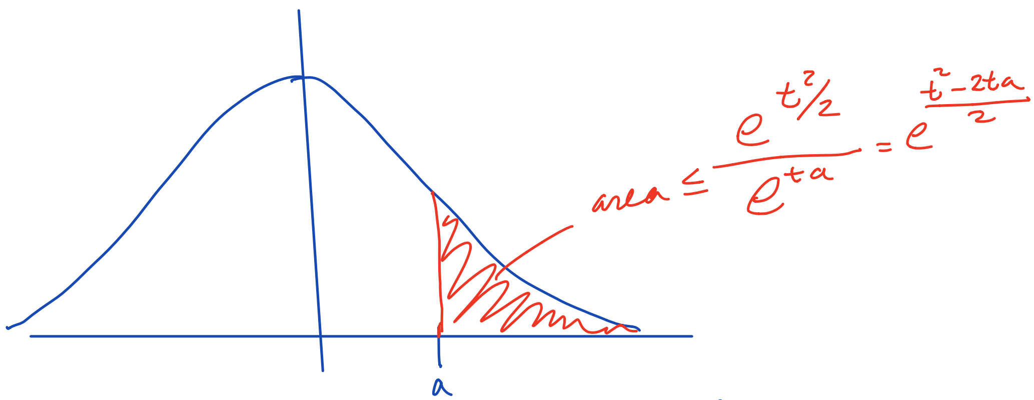
$$P(X \geq a) \leq e^{-at} M_X(t) \quad \forall t > 0.$$

Pf: The function $x \mapsto e^{xt}$ is monotone increasing for $t > 0$, so

$$P(X \geq a) = P(e^{tX} \geq e^{ta})$$

$$\leq \frac{E[e^{tX}]}{e^{ta}} = \frac{m_X(t)}{e^{ta}} \qquad \square$$

Ex: If $X = N(0,1)$, then



This also gives a lower bound for $\Phi(a)$.

$$1 - e^{\frac{t^2-2ta}{2}} \leq \Phi(a).$$